Cluster Algebras & g-vector fans

Running Examples

Example 1

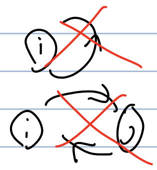
$$\tilde{B}_{t_0} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

$$\boxed{\hat{x}_3} \longrightarrow x_1 \longrightarrow x_2 \longleftarrow \boxed{\hat{x}_4}$$

Def (Ice quiver) finite directed graph Q

(i) $V_Q = \begin{Bmatrix} \text{mutable} \\ \text{vertices} \end{Bmatrix} \sqcup \begin{Bmatrix} \text{Frozen} \\ \text{vertices} \end{Bmatrix}$

(ii) No self-loops
No (oriented) 2-cycles



Extended exchange matrix

$$\tilde{B}_Q = \left[\begin{array}{c} B_Q \\ \hline \\ \end{array}\right] \qquad b_{ij} = \# \text{ arrows } i \to j - \# \text{ arrows } j \to i$$

Def (Quiver mutation)

Let k be a mutable vertex

step 1)

$$\begin{tikzcd} & k \arrow[dr] & \\ i \arrow[ur] \arrow[rr] & & j \end{tikzcd}$$

for all tuples of arrows $i \to k$, $k \to j$

Step 2) $i \to k \longmapsto k \leftarrow i$
$j \leftarrow k \longmapsto j \to k$

step 3) Remove all 2-cycles

$Q' = \mu_k(Q)$ is Q mutated at vertex k.

$$b_{ij}' = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + \dfrac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} & \text{else} \end{cases}$$

Def (Seed) $t \in \mathbb{T}_n$ the infinite n-regular tree

$$\Sigma^t = \left((x_{1,t}, \ldots, x_{n,t}), (\tilde{x}_{1,t}, \ldots, \tilde{x}_{m,t}), \tilde{B}_{Q,t} \right)$$

Seed mutation: for k mutable

$$\mu_k(x_{j,t}) = \begin{cases} x_{j,t} & j \neq k \\ \dfrac{\prod\limits_{\substack{1 \le i \le m \\ b_{ik} > 0}} (x_{i,t})^{b_{ik}} + \prod\limits_{\substack{1 \le i \le m \\ b_{ik} < 0}} (x_{i,t})^{-b_{ik}}}{x_{k,t}} \end{cases}$$

Example 1

$$\tilde{B}_{t_0} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right]$$

$$\begin{tikzcd} \boxed{\hat{x}_3} \arrow[r] & x_1 \arrow[r] & x_2 & \boxed{\hat{x}_4} \arrow[l] \end{tikzcd}$$

(x_1, x_2)

mutate at 1

$$\begin{tikzcd} \boxed{\hat{x}_3} \arrow[rr, bend right] & x_1 \arrow[l] & x_2 \arrow[l] & \boxed{\hat{x}_4} \arrow[l] \end{tikzcd}$$

$$\left(\frac{\hat{x}_3 + x_2}{x_1}, x_2 \right)$$

mutate at 2

$$\begin{tikzcd} \boxed{\hat{x}_3} & x_1 \arrow[r] & x_2 \arrow[r] \arrow[ll, bend left] & \boxed{\hat{x}_4} \arrow[ll, bend right] \end{tikzcd}$$

$$\left(\frac{\hat{x}_3 + x_2}{x_1}, \frac{\hat{x}_3 \hat{x}_4 + \frac{\hat{x}_3 + x_2}{x_1}}{x_2} \right) = \left(\frac{\hat{x}_3 + x_2}{x_1}, \frac{x_1 \hat{x}_3 \hat{x}_4 + \hat{x}_3 + x_2}{x_1 x_2} \right)$$

mutate at 1

$$\begin{tikzcd} \hat{x}_3 \arrow[rrr, bend left] & x_1 & x_2 \arrow[l] \arrow[ll, bend left] & \hat{x}_4 \end{tikzcd}$$

$$\left(\frac{x_1 \hat{x}_4 + 1}{x_2}, \frac{x_1 \hat{x}_3 \hat{x}_4 + \hat{x}_3 + x_2}{x_1 x_2} \right)$$

After μ_2 and then μ_1

you get back to where you

started (This is special b/c this example is of finite type)

Def (Cluster Algebra)

A commutative ring

$\mathcal{A} \subseteq \mathbb{C}(x_1, \dots, x_n, \hat{x}_1, \dots, \hat{x}_m)$

generated by all cluster variables

from all seeds over

ground ring $\mathbb{C}[\hat{x}_{n+1}^{\pm 1}, \dots, \hat{x}_m^{\pm 1}]$

Thm (Laurent Phenomenon)

$\forall$ cluster var $x_{i;t}$ $\quad x_{i;t}$ is a Laurent polynomial in $x_1, \dots, x_n$

Def (Cluster complex)

Abstract simplicial complex

ground set = {all cluster variables}

max simplices = clusters

Thm [Fomin–Zelevinsky] If $\mathcal{A}$ comes from a finite crystallographic (Φ, Δ)

$\exists$ bijection

$$\alpha = \sum_{i=1}^{n} d_i \alpha_i \longmapsto x[\alpha]$$

$$\alpha \in \Phi_+ \sqcup (-\Delta)$$

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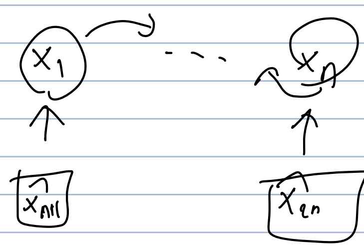
§2 F-polynomial & g-vectors

Def (Cluster Algebra w/ principal coefficients) $\mathcal{A}^{pr}$

$$\tilde{B}_{Q,t_0} = \left[\begin{array}{c} (B_{t_0})_{n\times n} \\ \hline Id_{n\times n} \end{array}\right]$$



Thm $\forall$ cluster variables $x_{i,t} = X^{pr}_{i,t}(x_1,\dots,x_n, x_{n+1},\dots,x_{2n}) \in \mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}, \hat{x}_{n+1},\dots,\hat{x}_{2n}]$

Def (F-polynomial)

$$F_{i,t}(y_1,\dots,y_n) := X^{pr}_{i,t}(1,\dots,1, y_1,\dots,y_n)$$

$$\tilde{B}_{t_0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{tikzcd} \boxed{\hat{x}_3} \arrow[r] & x_1 \arrow[r] & x_2 & \boxed{\hat{x}_4} \arrow[l] \end{tikzcd}$$

(x_1, x_2) F: $(1, 1)$

mutate at 1

$$\begin{tikzcd} \boxed{\hat{x}_3} \arrow[rr, bend right] & x_1 \arrow[l] & x_2 \arrow[l] & \boxed{\hat{x}_4} \arrow[l] \end{tikzcd}$$

$\left(\frac{\hat{x}_3 + x_2}{x_1}, x_2\right)$ F: $(1+y_1, 1)$

mutate at 2

$$\begin{tikzcd} \boxed{\hat{x}_3} & x_1 \arrow[r] & x_2 \arrow[r] \arrow[ll, bend left] & \boxed{\hat{x}_4} \arrow[ll, bend right] \end{tikzcd}$$

$$\left(\frac{\hat{x}_3 + x_2}{x_1}, \frac{\hat{x}_3\hat{x}_4 + \frac{\hat{x}_3 + x_2}{x_1}}{x_2}\right) = \left(\frac{\hat{x}_3 + x_2}{x_1}, \frac{x_1\hat{x}_3\hat{x}_4 + \hat{x}_3 + x_2}{x_1 x_2}\right)$$

F: $(1+y_1, y_1y_2 + y_1 + 1)$

mutate at 1

$$\begin{tikzcd} \boxed{\hat{x}_3} \arrow[rrr, bend left] & x_1 & x_2 \arrow[l] \arrow[ll, bend left] & \boxed{\hat{x}_4} \end{tikzcd}$$

$$\left(\frac{x_1 \hat{x}_4 + 1}{x_2}, \frac{x_1 \hat{x}_3 \hat{x}_4 + \hat{x}_3 + x_2}{x_1 x_2} \right)$$

Thm (Seperation of Additions)

$\mathcal{A}$: arbitrary cluster algebra

$\mathcal{A}^{pr}$: principal coefficients counter part

$x_{\ell,t}$ a cluster variable from $\mathcal{A}$

$$\Rightarrow \quad x_{\ell,t} = \frac{X_{\ell,t}(x_1,\dots,x_n, y_1,\dots,y_n)}{F^{\oplus}_{\ell,t}(y_1,\dots,y_n)}$$

where $y_j = \prod_{i=n+1}^{m} (\hat{x}_i)^{b_{ij}}$

$1 \le j \le n$

The y_j's live in $(\mathrm{Trop}(x_{n+1},\dots,x_m), \bullet, \oplus)$

multiplicative ab group on y_j's

$$\prod_{i=n+1}^{n} (\hat{x}_i)^{a_i} \oplus \prod_{i=n+1}^{n} (\hat{x}_i)^{b_i} := \prod_{i=n+1}^{n} (\hat{x}_i)^{\min(a_i, b_i)}$$

Example 2

$$\begin{tikzcd} & \boxed{\hat{x}_3} \arrow[dl, bend right] & \\ x_1 \arrow[r] & x_2 \arrow[u, bend right] & \boxed{\hat{x}_4} \arrow[l] \end{tikzcd}$$

$$\tilde{B}_{t_0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(Has principal part EX 1)

mutate at 2

$$\left(x_1, \ \frac{x_1 \hat{x}_4 + x_3}{x_2}\right)$$

From EX 1, mutation at 2 gave us

$$\left(\frac{x_1 \hat{x}_4 + 1}{x_2}, \ x_1\right)$$

F: $(1+y_2, 1)$

$$X = \frac{x_1 y_2 + 1}{x_2}$$

$F = 1 + y_2$

$$\frac{X}{F^{\oplus}} = \frac{\dfrac{x_1 (\hat{x}_3^{-1} \hat{x}_4) + 1}{x_2}}{1 \oplus (\hat{x}_3^{-1} \hat{x}_4)}$$

$$= \frac{\dfrac{x_1 (\hat{x}_3^{-1} \hat{x}_4) + 1}{x_2}}{\hat{x}_3^{-1}}$$

$$= \frac{x_1 x_4 + \hat{x}_3}{x}$$