Tropical linear spaces and the Bruhat-Tits building

Goal: Understand Tropical linear spaces as subcomplex of Bruhat-Tits building

$K = \mathbb{C}((t))$ $\mathrm{val}: K \to \mathbb{Z} \cup \{\infty\}$

$\mathcal{O} = \mathbb{C}[[t]]$ (power series)

$$M = k\left\{\overbrace{\begin{bmatrix} | & & | \\ f_1 & \cdots & f_n \\ | & & | \end{bmatrix}}^{n}\right. \in \mathrm{Mat}_{k \times n}(K) \quad \text{(full rank)}$$

Tropical plucker vector $P_I = \mathrm{val}(\det(f_{i_1}, \dots, f_{i_k}))$

$I = \{i_1, \dots, i_k\}$

Note: This is the basis version of the valuated matroid

If $\det = 0$ then $\mathrm{val} = \infty$

Tropical linear space: Fix a tropical plucker vector

$$L_p = \bigcap_{1 \le \tau_1 < \tau_2 < \dots < \tau_{k+1} \le n} \mathrm{Trop}\left(\sum_{r=1}^{k+1} P_{\tau \setminus \tau_r} x_{\tau_r}\right)$$

minimum of $\{P_{\tau \setminus \tau_r} + x_r\}$ is achieved twice

$\mathbb{TP}^{n-1} = \mathbb{R}^n / (1, \dots, 1)$

$\overline{\mathbb{TP}}^{n-1} = (\mathbb{R} \cup \infty)^n \setminus \{\infty, \dots, \infty\} / (1, \dots, 1)$

<u>Def</u> A subset $\Omega \subset \mathbb{TP}^{n-1}$ (or $\overline{\mathbb{TP}^{n-1}}$)

is tropically convex if it is

closed under (min, +) i.e.

If $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \Omega$
$\lambda, \mu \in \mathbb{R}$

then $(\min(x_1+\lambda, y_1+\mu), \dots, \min(x_n+\lambda, y_n+\mu)) \in \Omega$

[Yu-Yuster]

<u>Thm</u> L_p is the tropical convex hull in $\overline{\mathbb{TP}^{n-1}}$ of the

cocircuits $p(\sigma^*) := (p_{\sigma\cup 1}, p_{\sigma\cup 2}, \dots, p_{\sigma\cup n})$ $\quad \sigma \in \binom{[n]}{k-1}$

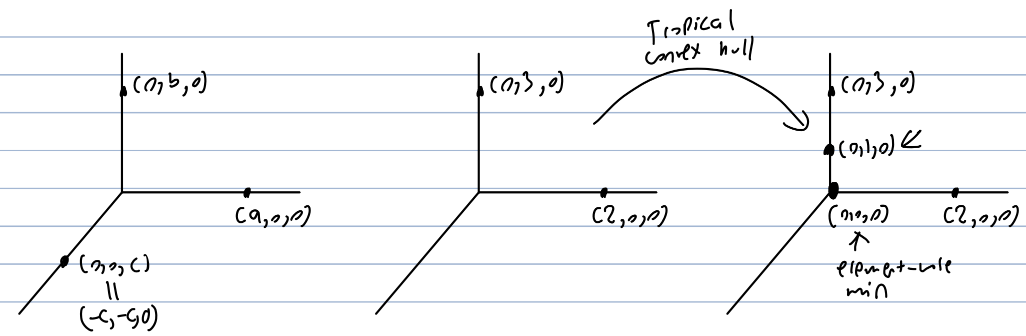
(If $|\sigma \cup i| \neq k$ then $p_{\sigma\cup i} = \infty$)

<u>EX</u> $k=2$ $\quad n=3$

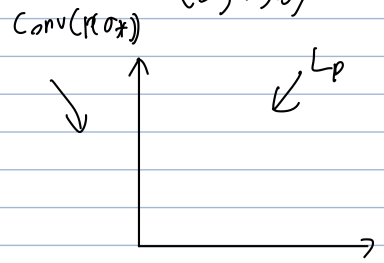
$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$p_I = 0 \quad \forall I$
(because valuation of a constant is 0)

L_p = min of x_1, x_2, x_3 is achieved twice



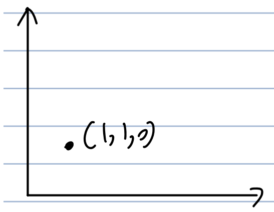
$\sigma = \{1\}$
$p(\sigma^*) = (p_{\sigma\cup 1}, p_{\sigma\cup 2}, p_{\sigma\cup 3})$
$= (\infty, 0, 0)$



Let Ω be tropically convex, $x \in \overline{\mathbb{TP}^{n-1}}$

Among all $w \in \Omega$ s.t. $w \geq x$ coordwise

$\exists!$ coord-wise min $\pi_\Omega(x)$. π_Ω is called the "nearest-point" map



$\Pi_R(1,1,0) = (0,0,0)$

Joswig – Sturmfels – Yu

$\Lambda \subset K^k$ a lattice
i.e., Λ is a free $\mathcal{O}$ submodule of full rank

$\Lambda = \mathrm{span}_{\mathcal{O}}(f_1, \dots, f_k)$ (Recall $\mathcal{O} = \mathbb{C}[[t]]$ has no neg powers so no neg valuations

$f_i \in K^k$

$\mathbb{B}$ has vertices {lattices}/$\sim$

$\Lambda \sim \Lambda' \iff \Lambda = c\Lambda'$ for some $c \in K^*$

edges in $\mathbb{B}$: $(\Lambda, \Lambda') \iff \exists t\ \ t\Lambda' \subset \Lambda \subset \Lambda'$

Make this into flag simplicial complex

For $k=2$ $\mathbb{B}$ is an infinite tree

For fixed lin ind $f_1, \dots, f_k$

consider all $\Lambda = \mathrm{span}_{\mathcal{O}}(t^{-a_1} f_1, \dots, t^{-a_k} f_k)$

This gives a subset of $\mathbb{B}$ called an apartment

If $k=2$ $\mathrm{span}_{\mathcal{O}}\{(t^{a} e_1, t^{-a_2} e_2) \mid (a,b) \in \mathbb{Z}^2\}$

$$\| $$

$\mathrm{span}_{\mathcal{O}}\{(e_1, t^{a-b} e_2) \mid (a,b) \in \mathbb{Z}^2/(1,1)\}$

Given $M = \begin{bmatrix} | & & | \\ f_1 & \cdots & f_n \\ | & & | \end{bmatrix}$ $\quad n \geq k$

let $[M] = \{ \Lambda = \mathrm{span}_{\mathcal{O}} (t^{-u_1} f_1, \ldots, t^{-u_n} f_n)$

$[M]$ is called a membrane

Thm [Keel–Tevelev]

The map $\Phi_M : \Lambda = \mathrm{span}_{\mathcal{O}}(t^{-u_1} f_1, \ldots) \longmapsto \pi_{L_p}(u_1, \ldots, u_n)$

is well-defined and induces an

isomorphism of simplicial complexes

between $[M]$ & L_p

EX

$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\qquad f_1 = e_1$
$f_2 = e_2$
$f_3 = e_1 + e_2$

$u = (1, 1, 0)$

$\Lambda = \mathrm{span}_{\mathcal{O}}(t^{-1} e_1, t^{-1} e_2, t^{-0}(e_1 + e_2))$

$= \mathrm{span}_{\mathcal{O}}(t^{-1} e_1, t^{-1} e_2)$

(As $t^{-0}(e_1 + e_2) = \underset{\uparrow \atop t \in \mathcal{O}}{t} \left(t^{-1} e_1 + t^{-1} e_2 \right)$